STRUCTURAL CAUSAL MODEL

Definition 7.1.1 (Causal Model)
A **causal model** is a triple

$$M = \langle U, V, F \rangle,$$

where:

(i) U is a set of **background** variables, (also called **exogenous**), that are determined by factors outside the model;

(ii) V is a set $\{V_1, V_2, \ldots, V_n\}$ of variables, called **endogenous**, that are determined by variables in the model—that is variables in $U \cup V$; and

(iii) F is a set of functions $\{f_1, f_2, \ldots, f_n\}$ such that each f_i is a mapping from (the respective domains of) $U \cup (V \setminus V_i)$ to V_i and such that the entire set F forms a mapping from U to V.
Interpretation on each f_i tells us the value of V_i given the values of all other variables in $U \cup V$, and the entire set F has a unique solution $V(u)$.
Symbolically, the set of equations F can be represented by writing

$$v_i = f_i(pa_i, u_i), \quad i = 1, \ldots, n,$$

where pa_i is any realization of a select set PA_i of variables in $V \setminus V_i$ (connoting *parents*) sufficient for representing f_i. Likewise, $U_i \subseteq U$ stands for a select set of variables in U sufficient for representing f_i.

SUBMODELS, ACTIONS, AND COUNTERFACTUALS

Definition 7.1.2 (Submodel)
Let M be a causal model, X a set of variables in V, and x a particular realization of X. A **submodel** M_x of M is the causal model

$$M_x = \langle U, V, F_x \rangle,$$

where

$$F_x = \{f_i : V_i \notin X\} \cup \{X = x\}. \qquad \textbf{(7.1)}$$

Definition 7.1.3 (Effect of Action)
Let M be a causal model, X be a set of variables in V, and x be a particular realization of X. The **effect of action** $do(X = x)$ on M is given by the submodel M_x.

Definition 7.1.4 (Potential Response)
Let X and Y be two subsets of variables in V. The potential response of Y to action $do(X = x)$, denoted $Y_x(u)$, is the solution for Y of the set of equations F_x.

Definition 7.1.5 (Counterfactual)
Let X and Y be two subsets of variables in V. The counterfactual sentence "The value that Y would have obtained, had X been x" is interpreted as denoting the potential response $Y_x(u)$.

PROBABILITIES OF COUNTERFACTUALS

- A **probabilistic causal model** is a pair $\langle M, P(u) \rangle$ where $P(u)$ assigns a probability to each state $U = u$.

- $P(u)$ induces unique distribution $P(v)$:

$$P(y) = \sum_{\{u|Y(u)=y\}} P(u)$$

- The probability of the counterfactual $Y_x = y$ is defined by submodel M_x:

$$P(Y_x = y) \triangleq \sum_{\{u|Y_x(u)=y\}} P(u) \triangleq P(y|do(x))$$

- The probability of joint-counterfactuals is well defined for any subsets X, Y, Z and W of V.

$$P(Y_x = y, Z_w = z) = \sum_{\{u|Y_x(u)=y, Z_w(u)=z\}} P(u)$$

CAUSAL THEORIES AND CAUSAL GRAPHS

Definition: A causal theory T is a partial specification of a causal model.

Alternatively: T is a set of models
e.g., T = causal graph G
e.g., $T = \langle G, P \rangle$

- **Causal graph** (defined by M) = G_M

1. (1) Draw arrow $V_i \rightarrow V_j$ iff $V_i \in PA_j$

2. (2) Draw bi-directed arc $V_i \leftarrow - - \rightarrow V_j$ unless there is a partition $\{S_i, S_j\}$ such that $U_i \subseteq S_i$), $U_j \subseteq S_j$ and $S_i \perp\!\!\!\perp S_j$ in $P(u)$.

EVALUATING CONDITIONAL COUNTERFACTUALS FROM A CAUSAL MODEL

Given: $M = \langle V, U, \{f_i\}, P(u) \rangle$

Query: find $P(Y_x = y | Z = z)$
(Z may be affected by X)

1. Update $P(u)$ by $Z = z$ $\quad P(u) \rightarrow P(u|z)$

2. Form the conditional model
$M^z = \langle V, U, \{f_i\}, P(u|z) \rangle$

3. Form the submodel $M^z_x = \langle V, U, F_x, P(u|z) \rangle$

4. Compute: $P(Y_x = y | Z = z) = P_{M^z_x}(Y = y)$

USED SELECTED SLIDES FROM IJCAI-99 PRESENTATION. THE COLOR VERSION OF THESE SLIDES CAN BE VIEWED AT

http://bayes.cs.ucla/IJCAI99/
(SLIDES #10–30 OF IJCAI-99 PRESENTATION)

Black and white .pdf file
will soon be available of these slides

AFTER SELECTED IJCAI-99 SLIDES, CONTINUE WITH SLIDE #7

READING INDEPENDECE OF COUNTERFACTUAL VARIABLES (ANOTHER USE OF TWIN NETWORKS)

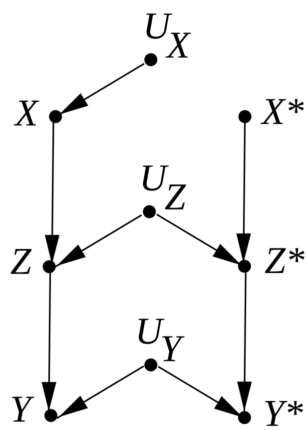

Figure 7.3: Twin network representation of the counterfactual Y_x in the model $X \to Z \to Y$.

Test if $(\boldsymbol{Y_x} \perp\!\!\!\perp \boldsymbol{X} | \boldsymbol{Z})$ holds in the chain model.

Similarly, $(Y_x \perp\!\!\!\perp X | Y, Z)$ is false, and $(Y_x \perp\!\!\!\perp X)$ is true.

More intricate tests:

$$Y_x \perp\!\!\!\perp X | \{Z, Z_x, Y\} \quad \text{and} \quad Y_x \perp\!\!\!\perp X | \{Y_z, Z_x, Y\}$$

because

$$(Y^* \perp\!\!\!\perp X | \{Z, U_Z, Y\})_G \text{ and } (Y^* \perp\!\!\!\perp X | \{U_Y, U_Z, Y\})_G$$

AXIOMS OF CAUSAL COUNTERFACTUALS

$Y_x(u) = y$: Y would be y, had X been x
(in state $U = u$)

1. **Definiteness**

$$\exists x \in X \ \ s.t. \quad X_y(u) = x$$

2. **Uniqueness**

$$(X_y(u) = x) \ \& \ (X_y(u) = x') \Longrightarrow x = x'$$

3. **Effectiveness**

$$X_{xw}(u) = x$$

4. **Composition**

$$W_x(u) = w \Longrightarrow Y_{xw}(u) = Y_x(u)$$

5. **Reversibility**

$$(Y_{xw}(u) = y) \ \& \ (W_{xy}(u) = w) \Longrightarrow Y_x(u) = y$$

SOUNDNESS AND COMPLETENESS

Theorem 7.3.3 (Soundness)
Composition, effectiveness, and reversibility are sound in structural model semantics; that is, they hold in all causal models.

Definition 7.3.4 (Recursiveness)
A model M is **recursive** if, for any two variables Y and W and for any set of variables X, we have

$$Y_{xw}(u) = Y_x(u) \quad \text{or} \quad W_{xy}(u) = W_x(u). \qquad \textbf{(7.24)}$$

Theorem 7.3.5 (Recursive Completeness)
Composition, effectiveness, and recursiveness are complete (Galles and Pearl 1998; Halpern 1998).

Theorem 7.3.6 (Completeness)
Composition, effectiveness, and reversibility are complete for all causal models (Halpern 1998).

CAUSAL EFFECTS FROM COUNTERFACTUAL LOGIC

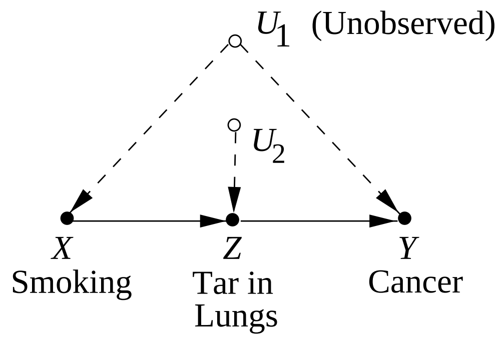

Figure 7.5

Conversion from graph to counterfactuals

Rule 1 (exclusion restrictions):

For every variable Y having parents PA_Y and for every set of variables $Z \subset V$ disjoint of PA_Y, we have

$$Y_{pa_Y}(u) = Y_{pa_Y z}(u). \qquad \textbf{(7.25)}$$

Rule 2 (independence restrictions):

If $Z_1, \ldots, Z_k$ is any set of nodes in V not connected to Y via paths containing only U variables, we have

$$Y_{pa_Y} \perp\!\!\!\perp \{Z_{1_{pa_{Z_1}}}, \ldots, Z_{k pa_{Z_k}}\}. \qquad \textbf{(7.26)}$$

Equivalently, (7.26) holds if the corresponding U terms $(U_{Z_1}, \ldots, U_{Z_k})$ are jointly independent of U_Y.

Example: $PA_X = \{\emptyset\}, PA_Y = \{Z\}$, and $PA_Z = \{X\}$.

$$Z_x(u) = Z_{yx}(u), \qquad \textbf{(7.27)}$$
$$X_y(u) = X_{zy}(u) = X_z(u) = X(u), \qquad \textbf{(7.28)}$$
$$Y_z(u) = Y_{zx}(u), \qquad \textbf{(7.29)}$$
$$Z_x \perp\!\!\!\perp \{Y_z, X\}. \qquad \textbf{(7.30)}$$

SYMBOLIC DERIVATION OF CAUSAL EFFECTS

$$\begin{aligned} Z_x(u) &= Z_{yx}(u), && \textbf{(7.27)} \\ X_y(u) &= X_{zy}(u) = X_z(u) = X(u), && \textbf{(7.28)} \\ Y_z(u) &= Y_{zx}(u), && \textbf{(7.29)} \\ Z_x &\perp\!\!\!\perp \{Y_z, X\}. && \textbf{(7.30)} \end{aligned}$$

Task 1

Compute $P(Z_x = z)$ (i.e., the causal effect of smoking on tar).

$$\begin{aligned} P(Z_x = z) &= P(Z_x = z|x) && \text{from (7.30)} \\ &= P(Z = z|x) && \text{by composition} \\ &= P(z|x). \end{aligned} \qquad \textbf{(7.31)}$$

Task 2

Compute $P(Y_z = y)$ (i.e., the causal effect of tar on cancer).

$$P(Y_z = y) = \sum_x P(Y_z = y|x)P(x). \qquad \textbf{(7.32)}$$

since (7.30) implies $Y_z \perp\!\!\!\perp Z_x | X$, we can write

$$\begin{aligned} P(Y_z = y|x) &= P(Y_z = y|x, Z_x = z) && \text{from (7.30)} \\ &= P(Y_z = y|x, z) && \text{by composition} \\ &= P(y|x, z). && \text{by composition} \end{aligned} \qquad \textbf{(7.33)}$$

Substituting (7.33) into (7.32) yields

$$P(Y_z = y) = \sum_x P(y|x, z)P(x). \qquad \textbf{(7.34)}$$

SYMBOLIC DERIVATION OF CAUSAL EFFECTS (Cont)

$$
\begin{aligned}
Z_x(u) &= Z_{yx}(u), && \textbf{(7.27)} \\
X_y(u) &= X_{zy}(u) = X_z(u) = X(u), && \textbf{(7.28)} \\
Y_z(u) &= Y_{zx}(u), && \textbf{(7.29)} \\
Z_x &\perp\!\!\!\perp \{Y_z, X\}. && \textbf{(7.30)}
\end{aligned}
$$

Task 3
Compute $P(Y_x = y)$ (i.e., the causal effect of smoking on cancer).

For any variable Z, by composition we have

$$Y_x(u) = Y_{xz}(u) \quad \text{if} \quad Z_x(u) = z.$$

Since $Y_{xz}(u) = Y_z(u)$ (from (7.29)),

$$Y_x(u) = Y_{xz_x}(u) = Y_z(u), \quad \text{where} \quad z_x = Z_x(u). \tag{7.35}$$

Thus,

$$
\begin{aligned}
P(Y_x = y) &= P(Y_{z_x} = y) && \text{from (7.35)} \\
&= \textstyle\sum_z P(Y_{z_x} = y | Z_x = z) P(Z_x = z) \\
&= \textstyle\sum_z P(Y_z = y | Z_x = z) P(Z_x = z) && \text{by composition} \\
&= \textstyle\sum_z P(Y_z = y) P(Z_x = z). && \text{from (7.30)}
\end{aligned} \tag{7.36}
$$

$$P(Y_x = y) = \sum_z P(z|x) \sum_{x'} P(y|z, x') P(x'). \tag{7.37}$$