11.3.3 Alternative Proof of the Back-Door Criterion

The original proof of the back-door criterion (Theorem 3.3.2) used an auxiliary intervention node F (Figure 3.2) and was rather indirect. An alternative proof is presented below, where the need for restricting Z to nondescendants of X is transparent.

Proof of the Back-Door Criterion

Consider a Markovian model G in which T stands for the set of parents of X. From equation (3.13), we know that the causal effect of X on Y is given by

$$P(y \mid \hat{x}) = \sum_{t \in T} P(y \mid x, t) P(t). \tag{11.6}$$

Now assume some members of T are unobserved. We seek another set Z of observed variables, to replace T so that

$$P(y \mid \hat{x}) = \sum_{z \in Z} P(y \mid x, z) P(z). \tag{11.7}$$

It is easily verified that (11.7) follow from (11.6) if Z satisfies:

(i) $(Y \perp\!\!\!\perp T \mid X, Z)$

(ii) $(X \perp\!\!\!\perp Z \mid T)$.

Indeed, conditioning on Z, (i) permits us to rewrite (11.6) as

$$P(y \mid \hat{x}) = \sum_{t} P(t) \sum_{z} P(y \mid z, x) P(z \mid t, x),$$

and (ii) further yields $P(z \mid t, x) = P(z \mid t)$, from which (11.7) follows.

It is now a purely graphical exercise to prove that the back-door criterion implies (i) and (ii). Indeed, (ii) follows directly from the fact that Z consists of nondescendants of X, while the blockage of all back-door paths by Z implies $(Y \perp\!\!\!\perp T \mid X, Z)_G$, hence (i). This follows from observing that any path from Y to T in G that is unblocked by $\{X, Z\}$ can be extended to a back-door path from Y to X, unblocked by Z.

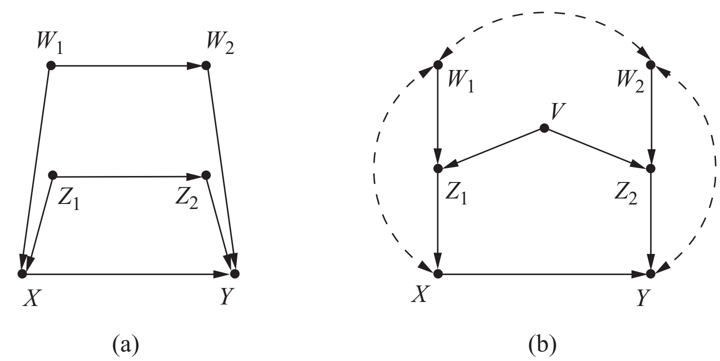

Figure 11.8 (a) $S_1 = \{Z_1, W_2\}$ and $S_2 = \{Z_2, W_1\}$ are each admissible yet not satisfying C_1 or C_2. (b) No subset of $C = \{Z_1, Z_2, W_1, W_2, V\}$ is admissible.

On Recognizing Admissible Sets of Deconfounders

Note that conditions (i) and (ii) allow us to recognize a set Z as *admissible* (i.e., satisfying equation (11.7)) starting from any other admissible T, not necessarily the parents of X. The parenthood status of T was used merely to established (11.6) but played no role in replacing T with Z to establish (11.7). Still, starting with the parent set T has the unique advantage of allowing us to recognize *every* other admissible set Z via (i) and (ii). For any other starting set, T, there exists an admissible Z that does not satisfy (i) and (ii). For an obvious example, choosing X's parents for Z would violate (i) and (ii) because no set can d-separate X from its parents as would be required by (i).

Note also that conditions (i) and (ii) are purely statistical, invoking no knowledge of the graph or any other causal assumption. It is interesting to ask, therefore, whether there are general independence conditions, similar to (i) and (ii), that connect any two admissible sets, S_1 and S_2. A partial answer is given by the Stone–Robins criterion (page 187) for the case where S_1 is a subset of S_2; another is provided by the following observation.

Define two subsets, S_1 and S_2, as *c-equivalent* ("c" connotes "confounding") if the following equality holds:

$$\sum_{s_1} P(y \mid x, s_1)P(s_1) = \sum_{s_2} P(y \mid x, s_2)P(s_2). \tag{11.8}$$

This equality guarantees that, if adjusted for, sets S_1 and S_2 would produce the same bias relative to estimating the causal effect of X on Y.

Claim: A sufficient condition for c-equivalence of S_1 and S_2 is that either one of the following two conditions holds:

$$C_1 : X \perp\!\!\!\perp S_2 \mid S_1 \quad \text{and} \quad Y \perp\!\!\!\perp S_1 \mid S_2, X$$
$$C_2 : X \perp\!\!\!\perp S_1 \mid S_2 \quad \text{and} \quad Y \perp\!\!\!\perp S_2 \mid S_1, X.$$

C_1 permits us to derive the right-hand side of equation (11.8) from the left-hand side, while C_2 permits us to go the other way around. Therefore, if S_1 is known to be admissible, the admissibility of S_2 can be confirmed by either C_1 or C_2. This broader condition allows us, for example, to certify $S_2 = PA_X$ as admissible from any other admissible set S_1, since condition C_2 would be satisfied by any such choice.

This broader condition still does not characterize *all* c-equivalent pairs, S_1 and S_2. For example, consider the graph in Figure 11.8(a), in which each of $S_1 = \{Z_1, W_2\}$ and

$S_2 = \{Z_2, W_2\}$ is admissible (by virtue of satisfying the back-door criterion), hence S_1 and S_2 are *c*-equivalent. Yet neither C_1 nor C_2 holds in this case.

A natural attempt would be to impose the condition that S_1 and S_2 each be *c*-equivalent to $S_1 \cup S_2$ and invoke the criterion of Stone (1993) and Robins (1997) for the required set-subset equivalence. The resulting criterion, while valid, is still not complete; there are cases where S_1 and S_2 are *c*-equivalent yet not *c*-equivalent to their union. A theorem by Pearl and Paz (2008) broadens this condition using irreducible sets.

Having given a conditional-independence characterization of *c*-equivalence does not solve, of course, the problem of identifying admissible sets; the latter is a causal notion and cannot be given statistical characterization.

The graph depicted in Figure 11.8(b) demonstrates the difficulties commonly faced by social and health scientists. Suppose our target is to estimate $P(y \mid do(x))$ given measurements on $\{X, Y, Z_1, Z_2, W_1, W_2, V\}$, but having no idea of the underlying graph structure. The conventional wisdom is to start with all available covariates $C = \{Z_1, Z_2, W_1, W_2, V\}$, and test if a proper subset of C would yield an equivalent estimand upon adjustment. Statistical methods for such reduction are described in Greenland et al. (1999b), Geng et al. (2002), and Wang et al. (2008). For example, $\{Z_1, V\}$, $\{Z_2, V\}$, or $\{Z_1, Z_2\}$ can be removed from C by successively applying conditions C_1 and C_2. This reduction method would produce three irreducible subsets, $\{Z_1, W_1, W_2\}$, $\{Z_2, W_1, W_2\}$, and $\{V, W_1, W_2\}$, all *c*-equivalent to the original covariate set C. However, none of these subsets is admissible for adjustment, because none (including C) satisfies the back-door criterion. While a theorem due to Tian et al. (1998) assures us that any *c*-equivalent subset of a set C can be reached from C by a step-at-a-time removal method, going through a sequence of *c*-equivalent subsets, the problem of covariate selection is that, lacking the graph structure, we do not know which (if any) of the many subsets of C is admissible. The next subsection discusses how external knowledge, as well as more refined analysis of the data at hand, can be brought to bear on the problem.